Title	Legendre-Galerkin Method for Poisson Equation on a Fan-Shaped Domain (Numerical Solution of Partial Differential Equations and Related Topics)
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Citation	数理解析研究所講究録 (2000), 1145: 161-169
Issue Date	2000-04
URL	http://hdl.handle.net/2433/63937
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
1145巻 2000年 161-169

Legendre-Galerkin Method for Poisson Equation on a Fan-Shaped Domain

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1 Introduction

In this paper we consider Poisson's equation with singularities caused by the presence of a corner. Singularities often occur in models of engineering problems due to discontinuities on the boundary conditions or abrupt changes in the boundary shape. When using numerical methods to solve problems with singularities, one must pay special attention to the singular domains. In both the finite difference and the finite element methods, local refinement is often employed near the singularity to obtain reasonable accuracy. However, it is generally difficult to discretize the equations defined on the domains with singularities by means of conventional numerical methods; moreover by those the error would propagate to the whole domain.

The Sinc-Galerkin method is well known as one of the most efficient methods to treat those problems[1]. It is particularly appealing because it can be used to solve the problems with boundary singularities, while maintaining its characteristic exponential convergence rate. The technique to combine the Sinc-Galerkin method and the domain decomposition method for solving the problems with singularities was proposed by Nancy J. Lybeck and Kenneth L.Bowers[4][5]. The Sinc-Galerkin domain decomposition methods can be applied to Poisson's equation on an L-shaped domain by dividing the underlying domain into some rectangular subdomains. However, since the restriction that the basic shape of subdomains is a rectangular domain , it is difficult to partition the domain with less than 90° interior angles.

The Sinc-Galerkin domain decomposition method, in which a fan-shaped domain is added as a basic shape, was proposed to make the way of partition more flexible. However, using the Sinc-Galerkin method it is difficult to determine h, which is the mesh size of each mesh point. Moreover, when solving the problems without singularities, too many mesh points near the boundaries cause a lot of waste or losing accuracy.

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In this paper we propose the Legendre-Galerkin method for solving Poisson's equation on a fan-shaped domain. Numerical results are presented to show the convergence of the approximate solutions.

2 Legendre-Galerkin Method

We consider Poisson's equation on the fan-shaped Ω given by

$$(1)\qquad \begin{cases} -\nabla^2 u(r,\theta) = f(r,\theta) & , (r,\theta) \in \Omega , \\ u(r,\theta) = b(\theta) & , (r,\theta) \in \gamma , \\ u(r,\theta) = 0 & , (r,\theta) \in \partial\Omega \backslash \gamma \end{cases}$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$, $f(r,\theta)$ is an analytic function of the two variables r and θ in a neighborhood of the point P and

$$(2)\qquad |u(r,\theta)| = O(r^{\frac{\pi}{\alpha}})$$

where $\frac{\pi}{\alpha} \neq m (m$ an integer) while

$$(3)\qquad |u(r,\theta)| = O(r^{\frac{\pi}{\alpha}} \log r)$$

when $\frac{\pi}{\alpha} = \frac{1}{m}(m = 1, 2, \cdots)$[7].

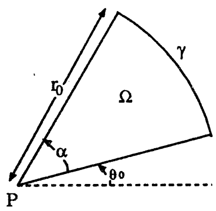

Figure 1: Fan-shaped domain Ω

2.1 Conformal Mapping

First, we remove the singularities at point P by using the transformation

$$(4)\qquad r = e^{-(\eta - \log r_0)}.$$

Then, Ω is mapped onto the semi-infinite strip domain $\tilde{\Omega}$ as shown in Fig 2. By this transformation, we obtain

$$(5)\qquad \begin{cases} -\tilde{\nabla}^2 \tilde{u}(\eta,\theta) = \tilde{f}(\eta,\theta) & , (\eta,\theta) \in \tilde{\Omega} , \\ \tilde{u}(\eta,\theta) = b(\theta) & , (\eta,\theta) \in \tilde{\gamma} , \\ \tilde{u}(\eta,\theta) = 0 & , (\eta,\theta) \in \partial\tilde{\Omega} \backslash \tilde{\gamma} \end{cases}$$

where $\tilde{\nabla}^2 = \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \theta^2}$.

Next, the choice of $\eta = \sinh^2(\rho)$ is made to transform $\tilde{u}(\eta,\theta)$ to $\tilde{u}(\sinh^2(\rho),\theta)$, which decays double exponentially with respect to ρ. Then, we find from (2),(3),(4) that

$$\max_{0\le\theta\le\alpha} |\tilde{u}(\sinh^2(\rho),\theta)| \;\cong\; 0 \;,\; \rho \ge L$$

for sufficiently large $L > 0$.

Therefore, since $\tilde{v}(\rho,\theta) = \tilde{u}(\sinh^2(\rho),\theta)$ can be regarded as a smooth even function with respect to ρ and be given by the cosine transform

$$(6) \quad \tilde{v}(\rho,\theta) \;=\; \frac{1}{2}c_0(\theta) + \sum_{k=1}^{\infty} c_k(\theta)\cos\frac{\pi}{L}k\rho \;,\quad c_k(\theta) \;=\; \frac{2}{L}\int_0^L \tilde{v}(\rho,\theta)\cos\frac{\pi}{L}k\rho d\rho$$

Thus, using the triangular polynomials, the approximation of $\tilde{v}(\rho,\theta)$ by the truncated series is accurate.

Furthermore, applying $\rho = \frac{L}{\pi}\cos^{-1} s$ to (6), we obtain

$$v(s,\theta) \;\equiv\; \tilde{v}(\frac{L}{\pi}\cos^{-1} s,\theta) \;=\; \frac{1}{2}c_0(\theta) + \sum_{k=1}^{\infty} c_k(\theta)T_k(s)$$

where $T_k(s)$ is the Chebyshev polynomial of degree k. Consequently, since $c_k(\theta), k \ge 0$ is analytic in $\theta \in [0,\alpha]$, we find that $v(s,\theta)$ can be well approximated by the polynomials in both s direction and θ direction.

Making a coordinate transformation

$$\begin{aligned} \eta = \psi_s(s) \;&=\; \sinh^2(\frac{L}{\pi}\cos^{-1} s) \;, \\ \theta = \psi_t(t) \;&=\; \frac{\alpha}{2}(t+1) \end{aligned}$$

in (5), $\tilde{\Omega}$ is mapped onto the square of domain $\hat{\Omega} = [-1,1]\times[-1,1]$ as shown in Fig 3.

Setting $v(s,t) = \tilde{u}(\psi_s(s),\psi_t(t))$, we get

$$(7) \qquad \begin{cases} -\dfrac{1}{\psi_s'}\dfrac{\partial}{\partial s}\left(\dfrac{1}{\psi_s'}\dfrac{\partial v}{\partial s}\right) - \dfrac{1}{\psi_t'}\dfrac{\partial}{\partial t}\left(\dfrac{1}{\psi_t'}\dfrac{\partial v}{\partial t}\right) = \hat{f}(s,t) \;,\; (s,t)\in\hat{\Omega}, \\ \qquad v(1,t) = \hat{b}(t) \;,\; t \in \hat{\gamma} \;, \\ \qquad v(s,t) = 0 \qquad ,\; t \in \hat{\Omega}\backslash\hat{\gamma} \end{cases}$$

where its solutions can be well approximated by the polynomial in the variable s and t.

In the computation, L is determined so that

$$\max_{0\le\theta\le\alpha} |\tilde{f}(\sinh^2(L),\theta)| \;\le\; \varepsilon_m$$

for sufficiently small ε_m.

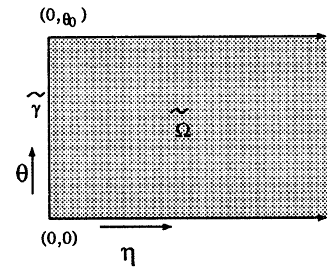

Fig 2: Semi-infinite strip domain $\tilde{\Omega}$

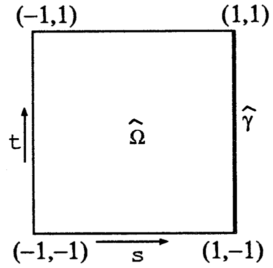

Fig 3: Square of domain $\hat{\Omega}$

2.2 Discretization

The approximate solution of (7) is given by

$$v_h(s,t) = \sum_{i=1}^{m+1}\sum_{j=1}^{n} v_{ij}\phi_i(s)\phi_j(t) \tag{8}$$

where m, n denotes the number of basis functions in the expansion. The basis functions used in solving (7) are

$$\begin{aligned}\phi_k(s) &= \frac{1}{\sqrt{4k+2}}(P_{k-1}(s) - P_{k+1}(s)) \ , \ k \neq m+1 \ , \\ \phi_{m+1}(s) &= \omega(s) = (s+1)^3\left(-\frac{3}{16}s + \frac{5}{16}\right)\end{aligned}$$

where $P_k(s)$ is the Legendre polynomial of degree k(See [3][5]).

The boundary basis functions $\omega(s)$ is necessary to satisfy the nonhomogeneous boundary conditions at the boundaries. This basis function was picked for its smoothness over the $\hat{\Omega}$ and its behavior at the boundary $\hat{\gamma}$. Here the basis function $\omega(s)$ is chosen to satisfy

$$\begin{cases} \omega(1) = 1 \\ \omega'(1) = 0 \\ \omega(-1) = \omega'(-1) = \omega''(-1) = 0 \end{cases} \tag{9}$$

Now we consider the nonhomogeneous boundary condition on $\hat{\gamma}$. We arrange the nodes of $\hat{\gamma}$ in an order according to the anticlockwise direction. Let the number of nodes be n and $\{t_j\}_{j=1}^n$ be the nodes on the boundary $\hat{\gamma}$. The nodal values on $\{t_j\}_{j=1}^n$ are also arranged in an order, which are denoted by $\hat{b}_1, \hat{b}_2, \cdots, \hat{b}_n$, and they form a n-dimensional column vector $\hat{\mathbf{b}} = (\hat{b}_1, \hat{b}_2, \cdots, \hat{b}_n)^T$, where by T we refer to the transpose of a matrix or a vector.

We obtain from (7),(8),(9) that

$$\hat{b}_l = v_h(1,t_l) = \sum_{j=1}^{n} v_{m+1,j}\phi_j(t_l) \ , \ 1 \leq l \leq n. \tag{10}$$

Let $\phi_j(t_l)$ be the entries of the matrix $\mathbf{H}$, (10) can be written in the form

$$\mathbf{H}^T\mathbf{v}_0 = \hat{\mathbf{b}} \tag{11}$$

where $\mathbf{v}_0 = (v_{m+1,1}, v_{m+1,2}, \cdots, v_{m+1,n})^T$. Therefore, we can obtain $\{v_{m+1,j}\}_{j=1}^n$ by solving (11).

Orthogonalizing the residual against basis functions $\phi_p(s)\phi_q(t)$ by using a inner product yields

$$\int_{\hat{\Omega}} \phi_p\phi_q \frac{\partial}{\partial s}\left(\frac{1}{\psi_s'}\frac{\partial v_h}{\partial s}\right)\psi_t' ds dt + \int_{\hat{\Omega}} \phi_p\phi_q \frac{\partial}{\partial t}\left(\frac{1}{\psi_t'}\frac{\partial v_h}{\partial t}\right)\psi_s' ds dt = \int_{\hat{\Omega}} \phi_p\phi_q\psi_s'\hat{f}\psi_t' ds dt. \tag{12}$$

Substituting (8) into (12), we obtain

$$\begin{aligned} \sum_{i=1}^{m+1}\sum_{j=1}^{n} v_{ij}\int_{-1}^{1}\phi_p'\frac{1}{\psi_s'}\phi_i' ds \int_{-1}^{1}\phi_j\psi_t'\phi_q dt &+ \sum_{i=1}^{m+1}\sum_{j=1}^{n} v_{ij}\int_{-1}^{1}\phi_p\psi_s'\phi_i ds \int_{-1}^{1}\phi_j'\frac{1}{\psi_t'}\phi_q' dt \\ &= \int_{\hat{\Omega}} \phi_p\phi_q\psi_s'\hat{f}\psi_t' ds dt. \end{aligned} \tag{13}$$

Setting $\mathbf{V} = [v_{ij}], 1 \le i \le m, 1 \le j \le n$, (13) can be written in the form

$$\mathbf{AVB} + \mathbf{CVD} = \mathbf{F} \tag{14}$$

where

$$\begin{aligned} \mathbf{A} &= \{a_{pi}\}_{p,i=1,2,\cdots,m}\ ,\ a_{pi} = \int_{-1}^{1}\phi_p'\frac{1}{\psi_s'}\phi_i' ds\ ,\ \mathbf{C} = \{c_{pi}\}_{p,i=1,2,\cdots,m}\ ,\ c_{pi} = \int_{-1}^{1}\phi_p\psi_s'\phi_i ds\ , \\ \mathbf{B} &= \{b_{jq}\}_{j,q=1,2,\cdots,n}\ ,\ b_{jq} = \int_{-1}^{1}\phi_j\psi_t'\phi_q dt\ \ ,\ \mathbf{D} = \{d_{jq}\}_{j,q=1,2,\cdots,n}\ ,\ d_{jq} = \int_{-1}^{1}\phi_j'\frac{1}{\psi_t'}\phi_q' dt. \end{aligned}$$

The entries $f_{pq}, 1 \le p \le m, 1 \le q \le n$ of the matrix $\mathbf{F}$ are given by

$$f_{pq} = \int_{\hat{\Omega}} \phi_p\phi_q\psi_s'\hat{f}\psi_t' ds dt - \sum_{j=1}^{n} v_{m+1,j}b_{jq}\int_{-1}^{1}\phi_p'\frac{1}{\psi_s'}\omega' ds - \sum_{j=1}^{n} v_{m+1,j}d_{jq}\int_{-1}^{1}\phi_p\psi_s'\omega ds.$$

a_{pi}, c_{pi}, f_{pq} can be given by using Gauss-type quadrature rules. On the other hand, using the orthogonality relation

$$\int_{-1}^{1} P_i(t)P_j(t)dt = \frac{2}{2j+1}\delta_{ij}$$

and the recurrence relation

$$(2j+1)P_j(t) = P_{j+1}'(t) - P_{j-1}'(t).$$

leads to

$$b_{ji} = b_{ij} = \begin{cases} c_ic_j\left(\frac{\alpha}{2j-1} + \frac{\alpha}{2j+3}\right) & i = j \\ -c_ic_j\frac{\alpha}{2i-1} & i = j+2 \\ 0 & others \end{cases}\ ,\ d_{ij} = \begin{cases} \frac{2}{\alpha} & i = j \\ 0 & i \ne j. \end{cases}$$

where $c_j = 1/\sqrt{4j+2}$.

We consider the following eigenvalue problem $(\alpha/2)\mathbf{B}\mathbf{x} = \lambda\mathbf{x}$ and let $\mathbf{\Lambda}$ be the diagonal matrix formed the eigenvalues and let $\mathbf{X}$ be the matrix formed by the corresponding eigenvectors. Then

$$\mathbf{BX} = \mathbf{DX\Lambda} = \frac{2}{\alpha}\mathbf{X\Lambda} \tag{15}$$

Making a change of variable $\mathbf{V} = \mathbf{W}\mathbf{X}^T$ in (14), we find

$$\mathbf{AWX}^T\mathbf{B} + \mathbf{CWX}^T\mathbf{D} = \mathbf{F}. \tag{16}$$

We then derive from (15) that

$$\mathbf{AW\Lambda} + \mathbf{CW} = \mathbf{G} \tag{17}$$

where $\mathbf{G} = (\alpha/2)\mathbf{FX}$. Let $\mathbf{w}_j = (w_{1j}, w_{2j}, \cdots, w_{mj})^T$ and $\mathbf{g}_j = (g_{1j}, g_{2j}, \cdots, g_{mj})^T$ for $j = 1, 2, \cdots, n$. Then the jth column of equation (17) can be written as

$$(\lambda_j\mathbf{A} + \mathbf{C})\mathbf{w}_j = \mathbf{g}_j. \tag{18}$$

In summary, the solution of (14) consists of the following step:

1. Compute the eigenvalues and eigenector s of $\mathbf{B}$
2. Compute $\mathbf{G}$
3. Obtain $\mathbf{W}$ by solving (18)
4. Set $\mathbf{V} = \mathbf{WX}^T$.

Since $\mathbf{B}$ can be split into two symmetric tridiagonal submatrices, the eigenvalues and eigenvectors of $\mathbf{B}$ can be easily computed in $O(n^2)$. Steps 2,3,4 take $O(mn^2)$, $O(nm^3)$, $O(mn^2)$, respectively. Therefore, the system (14) can be solved in about $O(mn^2 + nm^3)$.

3 Numerical Experiments

In this section, we present some numerical experiments. All computations are performed in double precistion on the Sun UltraSPARC2 workstation.

Consider the problem on the fan-shaped domain Ω with a interior angle α

$$\begin{aligned} -\nabla^2 u(x,y) &= f(x,y) \ , \ (x,y) \in \Omega \ , \\ u(x,y) &= 0 \qquad\ , \ (x,y) \in \partial\Omega \end{aligned}$$

where $f(x,y)$ is chosen so that the exact solution is given by

$$u(r,\theta) = (1-r)r^{\pi/\alpha}\sin(\pi\theta/\alpha) \ , \ (x,y) = (r\cos\theta, r\sin\theta).$$

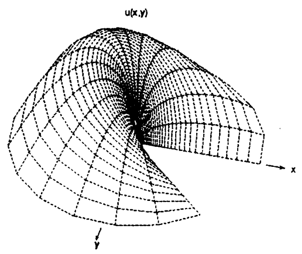

Fig 4: Solution for $\alpha = 7\pi/4$

Let $u_h(x, y)$ be the approximate solution on the domain Ω. We define the Legendre-Galerkin error norm by

$$\| E_U \| = \max_{(x,y)\in U} |u(x,y) - u_h(x,y)| \tag{19}$$

where

$$U = \{(0.9^k \cos(j\alpha/50), 0.9^k \sin(j\alpha/50)) : 0 \le k \le 50, 0 \le j \le 50\}.$$

We set $\alpha = 7\pi/4$ and $m = an$. A mesh plot of the approximate solution is shown in Fig 4. Comparing the numerical solutions with the exact solutions by (19), we obtain the approximate error norm as shown in Fig 5.

Let us make an analysis of the result shown in Fig 5. From the result the approximate error norm observed to be declining with increasing the values of a, but if a is more than $5/2$, the increasing of a has no significant influence to the result. Then, we can conclude that the values of n and m should be adjusted to each other for reasonable accuracy. In addition, as shown in Fig 6, when a is more than $5/2$, the Legendre-Galerkin method has the exponential convergence rate.

When setting $m = 3n$, we can obtain reasonable accuracy although the angle α becomes bigger as shown in Fig 6.

4 Conclusion

We proposed the Legendre-Galerkin method for solving Poisson'equation on a fan-shaped domain. Our proposed method leads to high accuracy and in conjunction with the domain decomposition method it can be applied to solve the problems with singularities caused by the presence of corners.

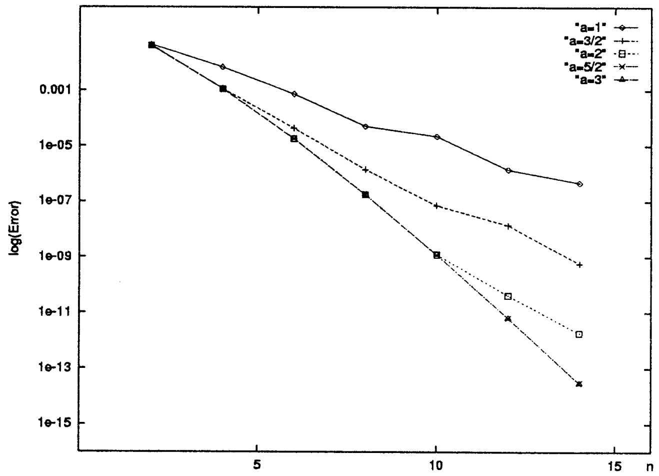

Fig 5: Convergence rates with n

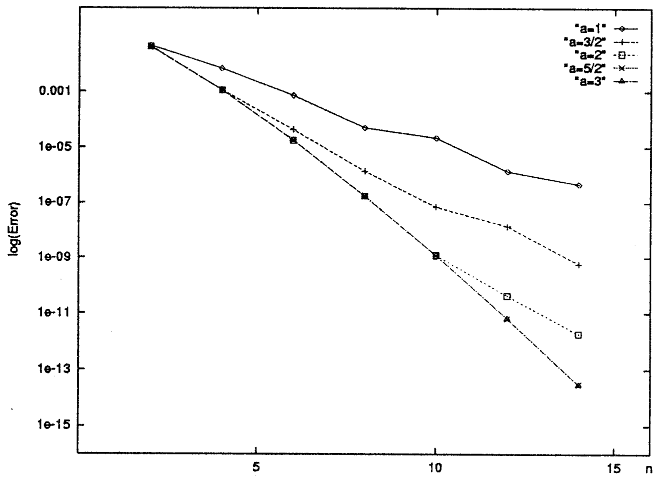

Fig 6: Convergence rates with α

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